ELSEVIER

Bull. Sci. math. 135 (2011) 1011–1013

BULLETIN DES SCIENCES MATHÉMATIQUES

www.elsevier.com/locate/bulsci

Erratum

Erratum to “Separation of representations with quadratic overgroups” [Bull. Sci. Math. 135 (2) (2011) 141–165]

Didier Arnal [a,*], Mohamed Selmi [b], Amel Zergane [a,b]

[a] *Institut de Mathématiques de Bourgogne, UMR CNRS 5584, Université de Bourgogne, U.F.R. Sciences et Techniques, B.P. 47870, F-21078 Dijon Cedex, France*

[b] *Département de Mathématiques, Unité de Recherche Physique Mathématique, Université de Sousse, Ecole Supérieure des Sciences et de Technologie de Hammam Sousse, Rue Lamine Abassi, 4011 H. Sousse, Tunisia*

Abstract

In the paper entitled “Separation of representations with quadratic overgroups”, we defined the notion of quadratic overgroups, and announced that the 6-dimensional nilpotent Lie algebra $\mathfrak{g}_{6,20}$ admits such a quadratic overgroup. There is a mistake in the proof. The present Erratum explains that the proposed overgroup is only weakly quadratic, and $\mathfrak{g}_{6,20}$ does not admit any natural quadratic overgroup.

1. Quadratic and weakly quadratic overgroups

Let us recall the definition of a quadratic overgroup for a Lie algebra $\mathfrak{g}$, given in [1], in the case where $\mathfrak{g}$ is nilpotent. We moreover add the notion of a weakly quadratic overgroup for $\mathfrak{g}$:

Definition 1.1. A quadratic overgroup (respectively a weakly quadratic overgroup), for the nilpotent algebra $\mathfrak{g}$, is a pair $(\mathfrak{g}^+, \Phi)$ where $\mathfrak{g}^+$ is a Lie algebra containing $\mathfrak{g}$ as a subalgebra, Φ is a polynomial mapping from $\mathfrak{g}^\star$ to $(\mathfrak{g}^+)^\star$, with degree at most two, and, for any pair of generic coadjoint orbits $\mathcal{O}$, $\mathcal{O}'$ in $\mathfrak{g}^\star$, $\overline{\mathrm{Conv}}(\Phi(\mathcal{O})) = \overline{\mathrm{Conv}}(\Phi(\mathcal{O}'))$ (respectively $\mathrm{Conv}(\Phi(\mathcal{O})) = \mathrm{Conv}(\Phi(\mathcal{O}')))$ implies $\mathcal{O} = \mathcal{O}'$.

DOI of original article: 10.1016/j.bulsci.2010.12.001.
* Corresponding author.
E-mail addresses: Didier.Arnal@u-bourgogne.fr (D. Arnal), Mohamed.Selmi@fss.rnu.tn (M. Selmi), amel.zergane@u-bourgogne.fr (A. Zergane).

0007-4497/$ – see front matter
doi:10.1016/j.bulsci.2011.11.001

In the paper [1], we prove that any nilpotent Lie algebra, with dimension less or equal to 6, admits a quadratic overgroup, except the Lie algebra $\mathfrak{g}_{6,20}$ whose commutation relations are:

$$[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4, \qquad [X_1, X_4] = X_5,$$
$$[X_2, X_3] = X_5, \qquad [X_2, X_5] = X_6, \qquad [X_3, X_4] = -X_6.$$

For this Lie algebra, we proposed an overgroup with Lie algebra $\mathfrak{g}^+ = \mathfrak{g}_{6,20} \ltimes S^2(\mathfrak{a})$, where $\mathfrak{a}$ is the abelian ideal $\mathfrak{a} = \mathrm{Vect}(X_6, X_5, X_4)$. Unfortunately, the argument in the proof does not hold. But we have

Lemma 1.2. *The pair* $(\mathfrak{g}^+, \Phi)$ *where* $\mathfrak{g}^+ = \mathfrak{g}_{6,20} \ltimes S^2(\mathfrak{a})$ *and* $\Phi(\ell) = (\ell, \ell^2|_{S^2(\mathfrak{a})})$*, proposed in* [1]*, is a weakly quadratic overgroup for* $\mathfrak{g}_{6,20}$.

Proof. Let ℓ be in $\mathfrak{g}^\star_{6,20}$, put $x_j = \ell(X_j)$. We say that ℓ is generic if $x_6 \neq 0$. To parametrize the coadjoint orbits in the set Ω of generic ℓ, we use the Vergne method [2]. We get that the orbit is characterized by two invariant functions λ and μ, the point in the orbit is determined by 4 real numbers denoted by p_k, q_k $(k = 1, 2)$ as follows:

$$\ell = (x_6, x_5, x_4, x_3, x_2, x_1) = \left(\lambda, \lambda q_2, -\lambda q_1, p_1 + \lambda \frac{q_2^2}{2}, p_2, \mu - \frac{\lambda}{6} q_2^3 + \frac{\lambda}{2} q_1^2 - q_2 p_1\right).$$

Therefore, a cross-section for the coadjoint action in Ω is given by the set:

$$\Sigma = \bigl\{\ell_0 = (\lambda, 0, 0, 0, 0, \mu),\ \lambda \in \mathbb{R}^\times,\ \mu \in \mathbb{R}\bigr\}.$$

The mapping Φ defined in [1] can be written as $\Phi(\ell) = (\ell, \varphi(\ell))$ where:

$$\varphi(\ell) = \left(x_6^2, x_6x_5, x_6x_4, x_5^2, x_5x_4, x_4^2\right) = \left(\lambda^2, \lambda^2 q_2, -\lambda^2 q_1, \lambda^2 q_2^2, -\lambda^2 q_1 q_2, \lambda^2 q_1^2\right).$$

Fix ℓ_0 in Σ, denote by $\mathcal{O}$ the coadjoint orbit passing through ℓ_0, then the result is a consequence of the relation:

$$\mathrm{Conv}\bigl(\Phi(\mathcal{O})\bigr) \cap \Phi(\Sigma) = \bigl\{\Phi(\ell_0)\bigr\}.$$

In fact, if $t_1, \ldots, t_r$ are real numbers in $]0, 1[$, such that $\sum t_j = 1$, and if p_{1j}, q_{1j}, p_{2j}, q_{2j} are parameters for points ℓ_j in $\mathcal{O}$, then $\sum t_j \Phi(\ell_j)$ belongs to $\Phi(\Sigma)$ only if:

$$0 = \sum t_j q_{2j}^2, \qquad 0 = \sum t_j q_{1j}^2.$$

That means $q_{1j} = q_{2j} = 0$ for all j, and thus $\sum t_j \Phi(\ell_j) \in \Phi(\Sigma)$ implies $\sum t_j p_{1j} = \sum t_j p_{2j} = 0$, and $\sum t_j \Phi(\ell_j) = \Phi(\ell_0)$. □

2. Non-existence of natural quadratic overgroups

Let us say that a quadratic overgroup $(\mathfrak{g}^+, \Phi)$ is natural if there is a submodule V in $S^2(\mathfrak{g})$, such that, for the Poisson bracket $\{\ ,\ \}$ on $\mathfrak{g}^\star$, $\{V, V\} = 0$, and $\Phi(\ell) = (\ell, (\ell)^2|_V)$. In this case, for any orbit $\mathcal{O}$, $\Phi(\mathcal{O})$ is a coadjoint orbit in $(\mathfrak{g}^+)^\star$. A direct computation gives:

Lemma 2.1. *Let* V *be a submodule in* $S^2(\mathfrak{g}_{6,20})$*, such that* $\{V, V\} = 0$*, then*

$$V \subset W = \mathrm{Vect}\bigl(X_6^2, X_6X_5, X_6X_4, X_5^2, X_5X_4, X_4^2, X_6X_1 + X_5X_3\bigr).$$

Corollary 2.2. *There is no natural quadratic overgroup for* $\mathfrak{g}_{6,20}$.

Proof. Clearly, it is enough to prove this for the case $V = W$. We keep the preceding notations, fix $\ell_0 = (\lambda, 0, 0, 0, 0, \mu)$ in Σ, a in $\mathbb{R}$, choose ℓ_n in the orbit $\mathcal{O}$ passing through ℓ_0, with parameter $q_{1n} = p_{2n} = 0$, $q_{2n} = an$, $p_{1n} = -\frac{\lambda}{2}a^2n^2$, then:

$$\begin{aligned}&\left(1-\frac{1}{n^3}\right)\ell_0+\frac{1}{n^3}\ell_n\\&\quad=\left(\lambda,\frac{\lambda a}{n^2},0,\frac{\lambda a^2}{n},0,\mu+\frac{\lambda a^3}{3},\lambda^2,\frac{\lambda^2 a}{n^2},0,\frac{\lambda^2 a^2}{n},0,0,\lambda\mu+\frac{\lambda^2 a^3}{3}\right).\end{aligned}$$

Therefore, if n goes to infinity, the point:

$$\Phi\left(\ell_0'\right)=\left(\lambda,0,0,0,\mu+\frac{\lambda a^3}{3},\lambda^2,0,0,0,0,0,\lambda\left(\mu+\frac{\lambda a^3}{3}\right)\right)$$

is in $\overline{\mathrm{Conv}}(\Phi(\mathcal{O})) \cap \Phi(\Sigma)$, this proves that, for any ν, the closure of $\mathrm{Conv}(\Phi(\mathcal{O}))$ contains $\Phi(\mathcal{O}')$, where $\mathcal{O}'$ is the orbit passing through $\ell_0' = (\lambda, 0, 0, 0, \nu)$. $\square$

References

[1] D. Arnal, M. Selmi, A. Zergane, Separation of representations with quadratic overgroups, Bull. Sci. Math. 135 (2011) 141–165.

[2] M. Vergne, La structure de Poisson sur l'algèbre symétrique d'une algèbre de Lie nilpotente, Bull. Soc. Math. France 100 (1972) 301–335.